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Exact solutions of multi-component nonlinear Schrödinger and Klein–Gordon equations in two-dimensional space–time

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Abstract

We present some exact solutions of a multi-component nonlinear partial differential equation which unifies nonlinear Schrödinger and Klein–Gordon equations in the two-dimensional space–time.

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1. Introduction

In this paper we consider the following nonlinear partial differential equation for a $\boldsymbol{C}^N$-valued function:

$$\Psi(x,t) = (\Psi_1(x,t), \ldots, \Psi_N(x,t))$$

on the two-dimensional space–time $\boldsymbol{R}^2 = \{(x,t)|x, t \in \boldsymbol{R}\}$ $(N \geqslant 1)$:

$$\mathrm{i}\alpha \frac{\partial \Psi(x,t)}{\partial t} + \beta \frac{\partial \Psi(x,t)}{\partial x} + \gamma \frac{\partial^2 \Psi(x,t)}{\partial t^2} + \frac{\partial^2 \Psi(x,t)}{\partial x^2} + \rho \Psi(x,t) + \kappa |\Psi(x,t)|^{2p} \Psi(x,t) = 0 \tag{1.1}$$

where $\alpha, \beta, \gamma, \rho, \kappa \in \boldsymbol{C}$ $(\kappa \neq 0)$, $p \in \boldsymbol{R} \setminus \{0\}$ (not necessarily an integer) are constants and $|\Psi(x,t)| := \sqrt{\sum_{n=1}^N |\Psi_n(x,t)|^2}$. Equation (1.1) unifies N-component nonlinear Schrödinger and Klein–Gordon equations on $\boldsymbol{R}^2$. The basic idea of the method taken in the present paper comes from a paper [4] which discusses a use of supersymmetric quantum mechanics in constructing soliton-type solutions to a multi-component nonlinear Schrödinger equation on $\boldsymbol{R}^2$

$$\mathrm{i}\frac{\partial \Phi(x,t)}{\partial t} + \frac{\partial^2 \Phi(x,t)}{\partial x^2} + \kappa |\Phi(x,t)|^2 \Phi(x,t) = 0 \tag{1.2}$$

in the case of (1.1) with $p = 1, \alpha = 1, \beta = 0, \gamma = 0, \rho = 0$. We pursue this method to find exact solutions of (1.1). In this paper we present exact solutions of (1.1) in the following cases: (i) $N = 1$, $p \in \boldsymbol{R} \setminus \{0, -1\}$ arbitrary; (ii) $N = 2$, $p = 1$; (iii) $N = 3$, $p = 1$.

2. Preliminaries

We seek solutions $\Psi = (\Psi_1, \ldots, \Psi_N)$ of (1.1) in the form of a travelling wave

$$\Psi_n(x,t) = e^{i\theta_n(x,t)}\psi_n(x - vt) \qquad n = 1, \ldots, N \tag{2.1}$$

where $v \in \boldsymbol{R}$ is a constant,

$$\theta_n(x,t) := \mu_n x - \omega_n t \tag{2.2}$$

with $\mu_n, \omega_n \in \boldsymbol{R}$ being constants, and ψ_n is a twice continuously differentiable function on $\boldsymbol{R}$. We assume that μ_n and ω_n satisfy

$$\mu_n = \frac{1}{2i}(i\alpha v - 2i\omega_n v\gamma - \beta) \tag{2.3}$$

$$2\omega_n v \operatorname{Im}\gamma = v \operatorname{Im}\alpha + \operatorname{Re}\beta. \tag{2.4}$$

We set

$$a := 1 + \gamma v^2 \tag{2.5}$$

$$b_n := \alpha\omega_n - \gamma\omega_n^2 - \mu_n^2 + \rho + i\beta\mu_n. \tag{2.6}$$

We only consider the case $a \neq 0$.

The following lemma is easily shown.

Lemma 2.1. *The function $\Psi := (\Psi_1, \ldots, \Psi_N)$ with Ψ_n given by (2.1) ($n = 1, \ldots, N$) is a solution of (1.1) if and only if, for each $n = 1, \ldots, N$,*

$$a\frac{d^2\psi_n(x)}{dx^2} + \kappa\left(\sum_{j=1}^{N}|\psi_j(x)|^2\right)^p \psi_n(x) + b_n\psi_n(x) = 0. \tag{2.7}$$

By lemma 2.1, the problem is reduced to finding solutions $(\psi_1, \ldots, \psi_N)$ of equation (2.7).

3. The case $N = 1$

In this case we set

$$\theta(x) := \theta_1(x) \qquad b := b_1. \tag{3.1}$$

A well known exact solution of (2.7) in the case $N = 1$ and $p = 1$ is given by $f_1(x) := \sqrt{2a/\kappa}\,\operatorname{sech}(x)$ with the condition $a + b = 0$, so that $F_1(x,t) := e^{i\theta(x,t)}f_1(x - vt)$ is a solution to equation (1.1) with $N = 1$, $p = 1$ and $a + b = 0$. We present other exact solutions of equation (1.1).

We try to find solutions of (2.7) with $N = 1$ in the form

$$\psi(x) = ce^{\phi(x)} \tag{3.2}$$

where $c \neq 0$ is a constant. By direct computation, ψ is a solution of (2.7) with $N = 1$ if and only if

$$a\big(\phi''(x) + \phi'(x)^2\big) + \kappa|c|^{2p}e^{2p\phi(x)} + b = 0. \tag{3.3}$$

It is not so hard to find solutions of (3.3) [2, appendix]. We only present results on solutions of (3.3).

Let $c \neq 0$ be a complex constant and C be a real constant. We introduce a function g_p on $\boldsymbol{R}$ by

$$g_p(y) := \begin{cases} -\dfrac{\kappa|c|^{2p}}{a(1+p)}e^{2py} + Ce^{-2y} - \dfrac{b}{a} & p \neq -1 \\ \left(C - \dfrac{2\kappa}{|c|^2 a}y\right)e^{-2y} - \dfrac{b}{a} & p = -1, \quad y \in \boldsymbol{R}. \end{cases} \tag{3.4}$$

Proposition 3.1. *Let G_p be a primitive function of either $1/\sqrt{g_p}$ or $-1/\sqrt{g_p}$ on*

$$D_+ := \{y \in \boldsymbol{R} | g_p(y) > 0\} \tag{3.5}$$

and G_p^{-1} denote the inverse function of G_p.

(i) For every open interval $J \subset D_+$,

$$\Psi(x,t) := c\mathrm{e}^{\mathrm{i}\theta(x,t)}\mathrm{e}^{G_p^{-1}(x-vt)} \tag{3.6}$$

is a solution of equation (1.1) with $N = 1$ on $\{(x,t)|x - vt \in G_p(J)\}$.

(ii) Suppose that there exists an open interval $J \subset D_+$ such that $G_p(J) = (0, L)$ with $L > 0$ or $L = \infty$ and

$$G_p^{-1}(0) := \lim_{x\downarrow 0} G_p^{-1}(x) \text{ exists and } g_p(G_p^{-1}(0)) = 0. \tag{3.7}$$

Let

$$\phi_p(x) := \begin{cases} G_p^{-1}(x) & x \in [0, L) \\ G_p^{-1}(-x) & x \in (-L, 0). \end{cases} \tag{3.8}$$

Then

$$\Psi(x,t) := c\mathrm{e}^{\mathrm{i}\theta(x,t)}\mathrm{e}^{\phi_p(x-vt)} \tag{3.9}$$

is a solution to equation (1.1) with $N = 1$ on $\{(x,t)|x - vt \in (-L, L)\}$.

Proof. (i) It is straightforward to check that $\phi(x) = G_p^{-1}(x)$ is a solution of equation (3.3). (ii) Similar to part (i). ☐

Some solutions given in proposition 3.1 may be global in (x,t) and have explicit representations. To write down some of them, we recall *q-deformed hyperbolic functions* which were introduced in [1]:

$$\sinh_q x := \frac{\mathrm{e}^x - q\mathrm{e}^{-x}}{2} \qquad \cosh_q x := \frac{\mathrm{e}^x + q\mathrm{e}^{-x}}{2} \tag{3.10}$$

$$\tanh_q x := \frac{\sinh_q x}{\cosh_q x} \qquad \mathrm{sech}_q x := \frac{1}{\cosh_q x} \qquad x \in \boldsymbol{R} \tag{3.11}$$

where $q > 0$ is a deformation parameter. Note that, if $q \neq 1$, then $\sinh_q x$ is not odd and $\cosh_q x$ is not even:

$$\sinh_q(-x) = -q \sinh_{1/q} x \qquad \cosh_q(-x) = q \cosh_{1/q} x \qquad x \in \boldsymbol{R}. \tag{3.12}$$

The following formulae can be easily proven:

$$(\sinh_q x)' = \cosh_q x \tag{3.13}$$

$$(\cosh_q x)' = \sinh_q x \tag{3.14}$$

$$\cosh_q^2 x - \sinh_q^2 x = q \tag{3.15}$$

$$(\tanh_q x)' = q\, \mathrm{sech}_q^2 x \tag{3.16}$$

$$(\mathrm{sech}_q x)' = -(\tanh_q x)(\mathrm{sech}_q x) \tag{3.17}$$

$$\tanh_q^2 x = 1 - q\, \mathrm{sech}_q^2 x. \tag{3.18}$$

Theorem 3.2. *Let $p \neq 0$. Suppose that $\kappa \in \boldsymbol{R}$ and $s \in \boldsymbol{R}$.*

$$\frac{a(1+p)}{\kappa} > 0 \qquad b + as^2 = 0. \tag{3.19}$$

Then the function

$$\Psi(x,t) = \left(\frac{a(1+p)qs^2}{\kappa}\right)^{1/2p} \mathrm{e}^{\mathrm{i}\theta(x,t)} \operatorname{sech}_q^{1/p}[sp(x-vt)] \tag{3.20}$$

is a solution of equation (1.1).

Theorem 3.3. *Let $b = 2as^2$ and $a/\kappa < 0$. Then*

$$\Psi(x,t) = -s\sqrt{-2a/\kappa}\,\mathrm{e}^{\mathrm{i}\theta(x,t)} \tanh_q s(x-vt) \tag{3.21}$$

is a solution of equation (1.1) with $N = 1$ and $p = 1$.

4. Exact solutions in the case $N \geqslant 2$

In this case, we follow an idea in [4]; namely, we try to find a potential $V : \boldsymbol{R} \to \boldsymbol{R}$ having the following properties (i)–(iii): (i) the one-dimensional Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + V$ admits N eigenfunctions $\psi_1, \ldots, \psi_N$ with eigenvalues $E_1, \ldots, E_N$ respectively:

$$-\frac{\mathrm{d}^2\psi_n(x)}{\mathrm{d}x^2} + V(x)\psi_n(x) = E_n\psi_n(x) \tag{4.1}$$

(ii) $K := b_n - aE_n$ is independent of $n = 1, \ldots, N$ and (iii) the eigenfunctions recover the potential V in the sense that

$$\kappa\left(\sum_{n=1}^{N} |\psi_n(x)|^2\right)^p + K = -aV(x). \tag{4.2}$$

If such a V exists, then $(\psi_1(x), \ldots, \psi_N(x))$ satisfies (2.7) and hence

$$\Psi(x) = \left(\mathrm{e}^{\mathrm{i}\theta_1(x,t)}\psi_1(x-vt), \ldots, \mathrm{e}^{\mathrm{i}\theta_N(x,t)}\psi_N(x-vt)\right) \tag{4.3}$$

is a solution of (1.1). Such potentials may be found in the class of the so-called shape-invariant potentials [1, 3, 5].

4.1. Shape-invariant potentials

For the reader's convenience we review basic general aspects of shape-invariant potentials. Let Λ be a subset of $\boldsymbol{R}$ and $\{W_\lambda\}_{\lambda\in\Lambda} \subset C^\infty(\boldsymbol{R} \to \boldsymbol{R})$. We introduce linear operators

$$A(\lambda) := -\frac{\mathrm{d}}{\mathrm{d}x} + W_\lambda \qquad A(\lambda)^+ := \frac{\mathrm{d}}{\mathrm{d}x} + W_\lambda \tag{4.4}$$

and define

$$H_+(\lambda) := A(\lambda)^+A(\lambda) \qquad H_-(\lambda) := A(\lambda)A(\lambda)^+. \tag{4.5}$$

We have

$$H_\pm(\lambda) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\lambda^\pm \tag{4.6}$$

where

$$V_\lambda^\pm := W_\lambda^2 \pm W_\lambda'. \tag{4.7}$$

In the context of supersymmetric quantum mechanics [6], the function W_λ and the pair $(H_+(\lambda), H_-(\lambda))$ are called a superpotential and a supersymmetric Hamiltonian respectively.

We assume the following hypothesis.

Hypothesis ($\boldsymbol{W}$). There exist mappings $f : \Lambda \to \Lambda$ and $F : f(\Lambda) \to \boldsymbol{R}$ such that for all $\lambda \in \Lambda$

$$V^+_{f(\lambda)} + F(f(\lambda)) = V^-_\lambda. \tag{4.8}$$

Remark 4.1. The functions $V^\pm_\lambda$ satisfying (4.8) are called *shape-invariant potentials*. This notion was first introduced by Gendenshteĭn [3] and developed by many theoretical physicists (see, e.g., [5]). The abstract mathematical formulation extending the idea of shape-invariant potentials was given in [1].

We write as $f^0(\lambda) := \lambda,\ f^n(\lambda) := f(f^{n-1}(\lambda)),\ n \geqslant 1$.

Lemma 4.1. *Assume* (W). *Let*

$$E_1(\lambda) := 0 \tag{4.9}$$

$$E_n(\lambda) := \sum_{j=1}^{n-1} F(f^j(\lambda)) \qquad n \geqslant 2 \tag{4.10}$$

$$\psi_{1,\lambda}(x) := \mathrm{e}^{\int_0^x W_\lambda(y)\,\mathrm{d}y} \tag{4.11}$$

$$\psi_{n,\lambda} := A(\lambda)^+ A(f(\lambda))^+ \cdots A(f^{n-2}(\lambda))^+ \psi_{1,f^{n-1}(\lambda)} \qquad n \geqslant 2. \tag{4.12}$$

Then, for all $\lambda \in \Lambda$,

$$H_+(\lambda)\psi_{n,\lambda} = E_n(\lambda)\psi_{n,\lambda} \qquad n \geqslant 1. \tag{4.13}$$

Proof. We prove (4.13) by induction. It is easy to see that (4.13) holds for $n = 1$. Suppose that (4.13) holds for some n. We have

$$\begin{aligned} H_+(\lambda)\psi_{n+1,\lambda} &= A(\lambda)^+ H_-(\lambda) A(f(\lambda))^+ \cdots A(f^{n-1}(\lambda))^+ \psi_{1,f^n(\lambda)} \\ &= A(\lambda)^+ H_-(\lambda)\psi_{n,f(\lambda)}. \end{aligned} \tag{4.14}$$

By hypothesis (W), we have

$$H_-(\lambda) = H_+(f(\lambda)) + F(f(\lambda)). \tag{4.15}$$

Putting this equation into (4.14) and using the induction hypothesis (4.13), we have

$$H_+(\lambda)\psi_{n+1,\lambda} = [E_n(f(\lambda)) + F(f(\lambda))]\psi_{n+1,\lambda} = E_{n+1}(\lambda)\psi_{n+1,\lambda}.$$

Hence (4.13) also holds for $n + 1$. □

Lemma 4.1 implies that, under hypothesis (W), for all $n \geqslant 1$,

$$-\frac{\mathrm{d}^2\psi_{n,\lambda}}{\mathrm{d}x^2} + V^+_\lambda \psi_{n,\lambda} = E_n(\lambda)\psi_{n,\lambda}. \tag{4.16}$$

Thus we obtain the following proposition.

Proposition 4.2. *Let* $N \geqslant 2$ *be fixed. Assume* (W). *Suppose that*

$$aV^+_\lambda(x) + \kappa\left(\sum_{j=1}^N |c_j|^2 |\psi_{j,\lambda}(x)|^2\right)^p + K = 0 \qquad x \in \boldsymbol{R} \tag{4.17}$$

with c_j *being complex constants,*

$$K := b_n - aE_n(\lambda) \tag{4.18}$$

independently of $n = 1, \ldots, N$. *Then*

$$\Psi_\lambda(x,t) := \left(c_1 \mathrm{e}^{\mathrm{i}\theta_1(x,t)}\psi_{1,\lambda}(x - vt), \ldots, c_N \mathrm{e}^{\mathrm{i}\theta_N(x,t)}\psi_{N,\lambda}(x - vt)\right) \tag{4.19}$$

is a solution of (1.1).

4.2. Exact solutions in the case $N = 2$ and $p = 1$

Let $s \in \boldsymbol{R}$ and consider the case where the superpotential W_λ is given by

$$W_\lambda(x) := -\lambda \tanh_q(sx) \qquad \lambda \in \boldsymbol{R}. \tag{4.20}$$

Then the functions $V_\lambda^\pm$ defined by (4.7) take the form

$$V_\lambda^\pm = -\lambda(\lambda \pm s)q\,\mathrm{sech}_q^2(sx) + \lambda^2. \tag{4.21}$$

Let

$$f_s(\lambda) = \lambda - s \qquad F_s(\lambda) := (\lambda + s)^2 - \lambda^2 = 2\lambda s + s^2. \tag{4.22}$$

Then it is easy to see that

$$V_{f_s(\lambda)}^+ + F_s(f_s(\lambda)) = V_\lambda^-. \tag{4.23}$$

Hence, for each s, W_λ satisfies hypothesis (W) with $\Lambda = \boldsymbol{R}$, $F = F_s$ and $f = f_s$. Thus we can apply lemma 4.1 and proposition 4.2. To do this, however, we need to compute the left-hand side of (4.17) in the present case.

We only consider the simplest case $p = 1$ in nonlinearity. Let

$$L_{\lambda,s}^{(N)}(x) := aV_\lambda^+(x) + \kappa\left(\sum_{j=1}^{N} |c_j|^2|\psi_{j,\lambda}(x)|^2\right) + K \tag{4.24}$$

and

$$h := \frac{1+q}{2}. \tag{4.25}$$

In the present case, we see that

$$E_n(\lambda) = \sum_{j=1}^{n-1} F_s(f_s^j(\lambda)) = (n-1)s[2\lambda - (n-1)s] \tag{4.26}$$

$$\psi_{1,\lambda}(x) = h^{\lambda/s}\,\mathrm{sech}_q^{\lambda/s}(sx) \tag{4.27}$$

$$\psi_{2,\lambda}(x) = (s - 2\lambda)h^{(\lambda-s)/s}\tanh_q(sx)\,\mathrm{sech}_q^{(\lambda-s)/s}(sx). \tag{4.28}$$

Using this expression, we see that

$$\begin{aligned} L_{\lambda,s}^{(2)}(x) = {} & -a\lambda(\lambda+s)q\,\mathrm{sech}_q^2(sx) + a\lambda^2 + K \\ & +\kappa\left(|c_1|^2h^{2\lambda/s} - |c_2|^2(2\lambda-s)^2qh^{2(\lambda-s)/s}\right)\mathrm{sech}_q^{2\lambda/s}(sx) \\ & +\kappa|c_2|^2(2\lambda-s)^2h^{2(\lambda-s)/s}\,\mathrm{sech}_q^{2(\lambda-s)/s}(sx). \end{aligned} \tag{4.29}$$

There are two ways to have $L_{\lambda,s}^{(2)} = 0$. One of them is to take $s = \lambda$. Then $L_{\lambda,\lambda}^{(2)} = 0$ if and only if

$$K = -\lambda^2(a + \kappa|c_2|^2) \tag{4.30}$$

$$\kappa(|c_1|^2h^2 - |c_2|^2\lambda^2q) = 2a\lambda^2q. \tag{4.31}$$

Hence we only need take b_n $(n = 1, 2)$ as

$$b_1 = -\lambda^2(a + \kappa|c_2|^2) \tag{4.32}$$

$$b_2 = -\kappa|c_2|^2\lambda^2 \tag{4.33}$$

to have (4.18) for $N = 2$. Thus we obtain the following theorem.

Theorem 4.3. *Suppose that (4.31)–(4.33) hold. Then*

$$\Psi(x,t) = \left(c_1\mathrm{e}^{\mathrm{i}\theta_1(x,t)}h\,\mathrm{sech}_q\lambda(x - vt),\ -c_2\lambda\mathrm{e}^{\mathrm{i}\theta_2(x,t)}\tanh_q\lambda(x - vt)\right) \tag{4.34}$$

is a solution to equation (1.1) with $N = 2$ and $p = 1$.

The other way to have $L^{(2)}_{\lambda,s} = 0$ is to take $s = \lambda/2$. Let $\lambda \neq 0$. Then $L^{(2)}_{\lambda,\lambda/2} = 0$ if and only if

$$K = -a\lambda^2 \tag{4.35}$$
$$aq = \tfrac{3}{2}\kappa|c_2|^2h^2 \tag{4.36}$$
$$|c_1|^2h^2 - |c_2|^2\left(\frac{3\lambda}{2}\right)^2 q = 0. \tag{4.37}$$

In this case we only need to take b_n as

$$b_1 = -a\lambda^2 \qquad b_2 = -\frac{a\lambda^2}{4} \tag{4.38}$$

to have (4.18) for $N = 2$. Thus we obtain the following theorem.

Theorem 4.4. *Let $\lambda \neq 0$ and suppose that (4.36)–(4.38) hold. Then*

$$\Psi(x,t) = \left(c_1 e^{i\theta_1(x,t)} h^2 \operatorname{sech}_q^2 \frac{\lambda(x-vt)}{2}, -\frac{3}{2}\lambda h c_2 e^{i\theta_2(x,t)} \tanh_q \frac{\lambda(x-vt)}{2} \operatorname{sech}_q \frac{\lambda(x-vt)}{2}\right) \tag{4.39}$$

is a solution of equation (1.1) with $N = 2$ and $p = 1$.

4.3. Exact solutions in the case $N = 3$ and $p = 1$

We next consider the case $N = 3$ and $p = 1$. We have

$$\psi_{3,\lambda}(x) = (3s - 2\lambda)h^{(\lambda-2s)/s}\left\{q(2\lambda - s)\operatorname{sech}_q^{\lambda/s}(sx) - 2(\lambda - s)\operatorname{sech}_q^{(\lambda-2s)/s}(sx)\right\}. \tag{4.40}$$

Hence we obtain

$$\begin{aligned} L^{(3)}_{\lambda,s}(x) = {} & a\lambda^2 + K - a\lambda(\lambda+s)q\operatorname{sech}_q^2(sx) \\ & + \kappa\left\{|c_1|^2h^{2\lambda}/s - |c_2|^2(2\lambda - s)^2 q h^{2(\lambda-s)/s} \right. \\ & \left. + |c_3|^2 h^{2(\lambda-2s)/s} q^2(2\lambda - s)^2\right\}\operatorname{sech}_q^{2\lambda/s}(sx) \\ & + \kappa\left\{|c_2|^2(2\lambda - s)^2 h^{2(\lambda-s)/s} - 4|c_3|^2h^{2(\lambda-2s)/s}(2\lambda - s)(\lambda - s)\right\} \\ & \times\operatorname{sech}_q^{2(\lambda-s)/s}(sx) + 4\kappa|c_3|^2(\lambda - s)^2 h^{2(\lambda-2s)/s}\operatorname{sech}_q^{2(\lambda-2s)/s}(sx). \end{aligned} \tag{4.41}$$

As in the preceding case $N = 2$, there are two choices for s that give $L^{(3)}_{\lambda,s} = 0$. One is to take $s = \lambda$. In this case we obtain the following theorem.

Theorem 4.5. *Suppose that b_1 and b_2 are given by (4.32) and (4.33) respectively, $b_3 = b_1$ and*

$$\kappa(|c_1|^2h^2 - |c_2|^2\lambda^2 q + |c_3|^2h^{-2}q^2\lambda^2) = 2a\lambda^2 q. \tag{4.42}$$

Then

$$\begin{aligned} \Psi(x,t) = {} & \left(c_1 e^{i\theta_1(x,t)} h \operatorname{sech}_q \lambda(x-vt), -c_2\lambda e^{i\theta_2(x,t)} \tanh_q \lambda(x-vt), \right. \\ & \left. c_3 e^{i\theta_3(x,t)}\lambda^2 h^{-1} q \operatorname{sech}_q \lambda(x-vt)\right) \end{aligned} \tag{4.43}$$

is a solution of equation (1.1) with $N = 3$ and $p = 1$.

The other choice is to take $s = \lambda/3$. In this case we obtain the following theorem.

Theorem 4.6. *Suppose that*

$$b_1 = -a\lambda^2 \tag{4.44}$$

$$b_2 = -\tfrac{4}{9}a\lambda^2 \tag{4.45}$$

$$b_3 = -\tfrac{1}{9}a\lambda^2 \tag{4.46}$$

$$|c_1|^2h^4 - |c_2|^2(\tfrac{5}{3}\lambda)^2qh^2 + |c_3|^2q^2(\tfrac{5}{3}\lambda)^2 = 0 \tag{4.47}$$

$$|c_2|^2h^2 = \tfrac{8}{3}|c_3|^2q\lambda \tag{4.48}$$

$$aq = \tfrac{4}{3}\kappa|c_3|^2h^2. \tag{4.49}$$

Then

$$\Psi(x,t) = \Bigg(c_1\mathrm{e}^{\mathrm{i}\theta_1(x,t)}h^3\,\mathrm{sech}_q^3\frac{\lambda(x-vt)}{3},\; -\frac{5}{3}c_2\mathrm{e}^{\mathrm{i}\theta_2(x,t)}h^2\lambda\tanh_q\frac{\lambda(x-vt)}{3}\,\mathrm{sech}_q^2\frac{\lambda(x-vt)}{3},\; \frac{1}{3}\lambda^2hc_3\mathrm{e}^{\mathrm{i}\theta_3(x,t)}\left[5q\,\mathrm{sech}_q^3\frac{\lambda(x-vt)}{3} - 4\,\mathrm{sech}_q\frac{\lambda(x-vt)}{3}\right]\Bigg) \tag{4.50}$$

is a solution of equation (1.1) with $N = 3$ and $p = 1$.

In the same manner as above, one may continue to calculate $L_{\lambda,s}^{(N)}$ for $N \geqslant 4$ and check whether there exist constants c_j, $j = 1, \ldots, N$, such that $L_{\lambda,s}^{(N)} = 0$. It is an interesting problem to show whether or not, for *all* $N \geqslant 4$, there exist constants c_j, $j = 1, \ldots, N$ such that $L_{\lambda,s}^{(N)} = 0$, but this problem is left open.

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References

[1] Arai A 1991 Exactly solvable supersymmetric quantum mechanics *J. Math. Anal. Appl.* **158** 63–79

[2] Arai A 2000 Supersymmetric methods for constructing soliton-type solutions to multi-component nonlinear Schrödinger and Klein–Gordon equations *Hokkaido University Preprint Series in Mathematics* No 508

[3] Gendenshteîn L E 1983 Derivation of exact spectra of the Schrödinger equation by means of supersymmetry *JETP Lett.* **38** 356–9

[4] Hrubý J 1989 On SSQM and the $U(N)$ nonlinear Schrödinger equation *J. Phys. A: Math. Gen.* **22** 1807–18

[5] Lévai G 1989 A search for shape-invariant potentials *J. Phys. A: Math. Gen.* **22** 689–702

[6] Witten E 1981 Dynamical breaking of supersymmetry *Nucl. Phys.* B **185** 513–54